Modern Portfolio Theory Notes

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1 Introduction

This material can also be found in chapters six and seven of the textbook (but not as good).

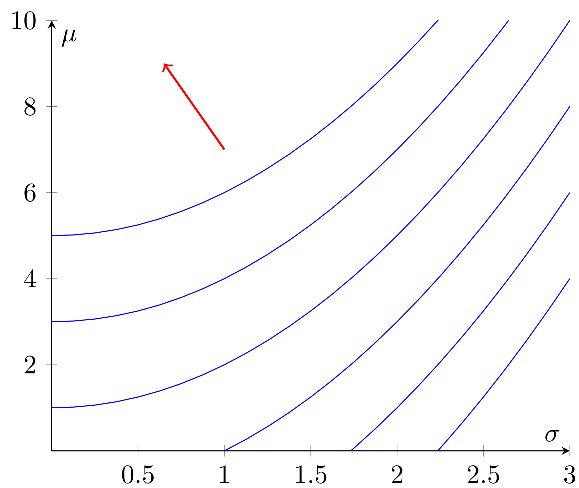
We will use the simplest utility function that captures the important features.

$$U(r) = \mathbb{E}[r] - \frac{A}{2}\sigma(r)^2$$

where r is the return on your wealth. In essence, you get higher utility from getting a higher return, but lower utility from higher variance. The A is a coefficient that represents your level of risk aversion. If A is positive, you are risk averse. If A is negative, you are risk loving. If A is zero, you are neutral. The higher the value of A, the more risk averse you are. We will typically think of A being positive.

If we were to draw a graph with expected return on the y-axis and standard deviation of the x-axis, what would your indifference curves look like? Can you explain why they should look like that?

Answer:



Let's try a numerical example. Consider three different portfolios with different expected returns and levels of risk. For investors with any given level of risk aversion, we can compute which portfolio is prefered.

Portfolio	Expected Return	Risk(SD)	Utility		
			$A = 2$	$A = 3.5$	$A = 5$
L (low risk)	7%	5%	.0675	.0656	.0638
M (medium risk)	9%	10%	.0800	**.0725**	**.0650**
H (high risk)	13%	20%	**.09**	.06	.03

Typically you will see a pattern similar to what is displayed in the table. Assets with a higher expected return will also have higher risk. Can you explain why that is?

We will often talk about a return by comparing it to the risk free return (Is there a risk free asset in the real world? What would it be?). I will define the **excess return** of an asset to be its return minus the risk free return. We will call the **risk premium** the expected value of the excess return.

$$\textit{risk premium}_i = \mathbb{E}[r_i] - r_f$$

Consider the portfolios in the above table. If the risk free rate was 5%, then the low, medium, and high risk portfolios would have risk premium of 2%, 4%, and 8% respectively.

2 One Risky Asset and One Risk Free Asset

Now that we can evaluate different assets or different portfolios, let's consider the problem of finding the optimal combinations or mixtures of assets.

Imagine that there is one risk free security with return denoted r_f and one risky security with return denoted r_r. If we know the mean an variance, we can determine which one the investor would rather put all their money in. However, in the real world you usually have more options than that. You could choose to put any fraction of your wealth in the risky asset and the remainder in the risk free asset.

Let α be the fraction of your wealth that you put in the risky asset. Now the return that you get is a combination of the two returns.

$$r_c = \alpha r_r + (1-\alpha) r_f$$

We can calculate the expected return and variance of any combination defined by α.

$$\begin{aligned}\mathbb{E}[r_c] &= \mathbb{E}[\alpha r_r + (1-\alpha) r_f] \\ &= \alpha \mathbb{E}[r_r] + (1-\alpha) r_f \\ &= r_f + \alpha(\mathbb{E}[r_r] - r_f)\end{aligned}$$

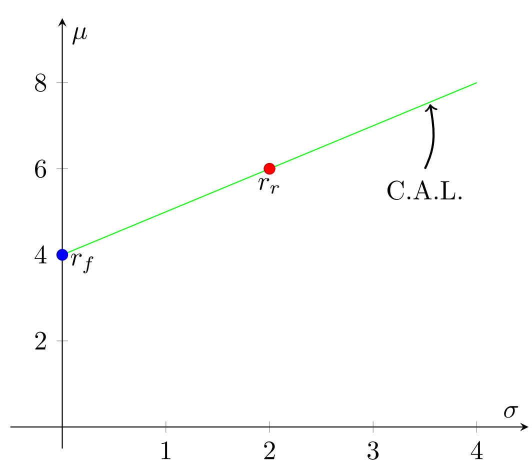
Notice that the expected return on the combination is simply the linear combination of the expected returns. The expected return on your combination will be equal to the risk free rate plus α times the risk premium for r_r.

Now let's try the variance.

$$\begin{aligned}V[r_c] &= V[\alpha r_r + (1-\alpha) r_f] \\ &= V[\alpha r_r] \\ &= \alpha^2 \sigma(r_r)^2 \\ \Rightarrow \sigma(r_c) &= \alpha \sigma(r_r)\end{aligned}$$

Since r_f is risk-free, it has zero variance. We see that the variance of the combination isn't just the combination of the variances (you need to square α), but the standard deviation is that simple.

Thus, if we were to draw on a mean-standard deviation graph all the possible portfolios, they would simply be a straight line through the risk free return and the risky return.



The green line, which represents every possible portfolio made up of the risk free security and the single risky security, is known as the **capital allocation line**.

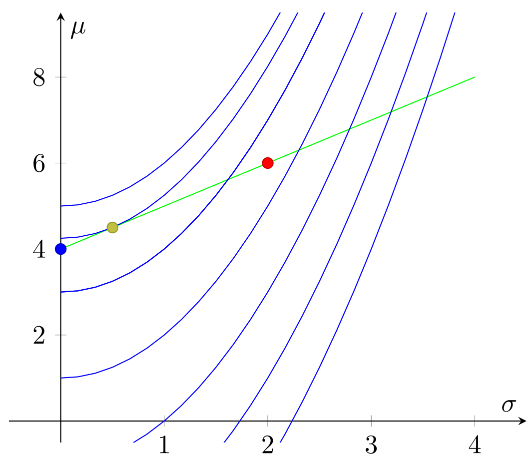
Notice that the slope of the capital allocation line is equal to the risk premium devided by the standard deviation of the risky asset. This fraction is known as the **Sharpe ratio**. We can now write the capital allocation line using the following formula.

$$\mathbb{E}[r_c] = r_f + \sigma_c \frac{\mathbb{E}[r_r] - r_f}{\sigma_r}$$

When written this way, we can more clearly see that the intercept is the risk free rate and the slope is the Sharpe ratio.

The consumer chooses α to maximize their utility subject to being on the capital allocation line.

The capital allocation line plays the role that the budget constraint played in your intro to economics class. We need to find the tangent indifference curve. (For practice, draw this same thing for an investor that is more risk averse).



We can solve analytically for that tangency point. The problem we need to solve is,

$$\max_{\alpha} \quad \mathbb{E}[r_c] - \frac{A}{2}\sigma(r_c)^2.$$

Remember that we can rewrite this as,

$$\max_{\alpha} \quad r_f + \alpha\left(\mathbb{E}[r_r] - r_f\right) - \frac{A}{2}\alpha^2\sigma(r_r)^2.$$

In order to maximize this, we must take the derivative and set it equal to zero.

$$\frac{\partial U}{\partial \alpha} = \mathbb{E}[r_r] - r_f - \alpha A \sigma(r_r)^2$$

This implies,

$$\alpha^* = \frac{\mathbb{E}[r_r] - r_f}{A\sigma^2}.$$

Notice that the amount of the risky asset you hold is increasing in the risk premium, decreasing in the variance, and decreasing in your level of risk aversion. All those are as you would likely expect.

3 Two Risky Assets

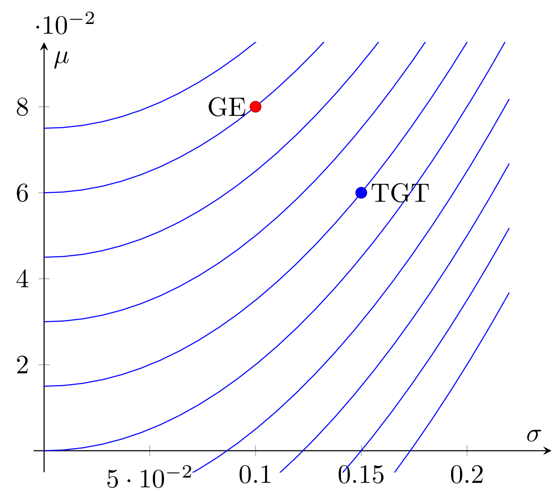
Now let's move on to a different problem. Suppose now that there are two risky assets in which you can invest, r_1 and r_2.

We will look at a numeric example to make sure we understand how the utility function works. Suppose that GE stock has an expected return of 8% and a variance of 12%. Also, suppose that Target stock has an expected return of 5% and a variance of 14%. Assume the two stocks are uncorrelated and that your coefficient of risk aversion is $A = 4$.

If given only the three choices, would you rather invest all your money in GE, all your money in Target, or half your money in each? It would appear at first glance that you should put all your money in GE. It has a better (higher) expected return, and it has a better (lower) variance. Those are the only two things you care about, so why would you put any money in Target? Let's just run the numbers quickly and see if this intuition is correct.

$$U(GE) = \mathbb{E}[GE] - \frac{4}{2}\sigma(GE)^2 = 0.08 - 2 * 0.12 = -0.16$$
$$U(Target) = \mathbb{E}[Target] - \frac{4}{2}\sigma(Target)^2 = 0.05 - 2 * 0.14 = -0.23$$

This confirms our intuition that GE stock is prefered to Target. We could have also seen this by graphing the two assets with indifference curves.



The last option we need to consider is an equal split of the two assets. Looking at the graph, is there any way a combination of the red dot and the blue dot can get us on a higher indifference curve? Let's put in the numbers and see.

$$\begin{aligned}
U\left(\frac{1}{2}GE + \frac{1}{2}Target\right) &= \mathbb{E}\left[\frac{1}{2}GE + \frac{1}{2}Target\right] - \frac{4}{2}\sigma\left(\frac{1}{2}GE + \frac{1}{2}Target\right) \\
&= \frac{1}{2}\mathbb{E}[GE] + \frac{1}{2}\mathbb{E}[Target] - \frac{4}{2}\frac{1}{4}\sigma(GE)^2 - \frac{4}{2}\frac{1}{4}\sigma(Target)^2 \\
&= \frac{1}{2} * 0.08 + \frac{1}{2} * 0.05 - \frac{1}{2} * 0.12 - \frac{1}{2} * 0.14 \\
&= -0.065
\end{aligned}$$

This does not match the earlier stated intuition. The mixture of half your money in GE and half in Target is strictly prefered to putting all your money in either one of them alone. This is the principle of **diversification**. Simply put, diversification if the old saying that "you shouldn't put all your eggs in one basket". In the graph, it must be that the set of possible portfolios is not simply a straight line between the two assets as it was in the last section.

Consider more generally, that there are two risky assets with returns r_1 and r_2. I will denote σ_1 and σ_2 the standard deviation of r_1 and r_2 respectively. Let ρ be the correlation between r_1 and r_2. As before, α will be the fraction of wealth put in asset 1.

The return on your portfolio is

$$r_c = \alpha r_1 + (1-\alpha) r_2.$$

The expectation of this is simply computed.

$$\begin{aligned}\mathbb{E}[r_c] &= \mathbb{E}[\alpha r_1 + (1-\alpha) r_2] \\ &= \alpha \mathbb{E}[r_1] + (1-\alpha)\mathbb{E}[r_2]\end{aligned}$$

The variance is only slightly more complicated.

$$\begin{aligned}V[r_c] &= V[\alpha r_1 + (1-\alpha) r_2] \\ &= V[\alpha r_1] + V[(1-\alpha) r_2] + 2Cov[\alpha r_1, (1-\alpha) r_2] \\ &= \alpha^2 V[r_1] + (1-\alpha)^2 V[r_2] + 2\alpha(1-\alpha) Cov[r_1, r_2] \\ &= \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2\rho\end{aligned}$$

Let's look at a few special cases to get some intuition for this equation. Notice that the variance is increasing in ρ. Also, remember that $\rho \in [-1, 1]$. So, let's start with the worst case scenario where $\rho = 1$.

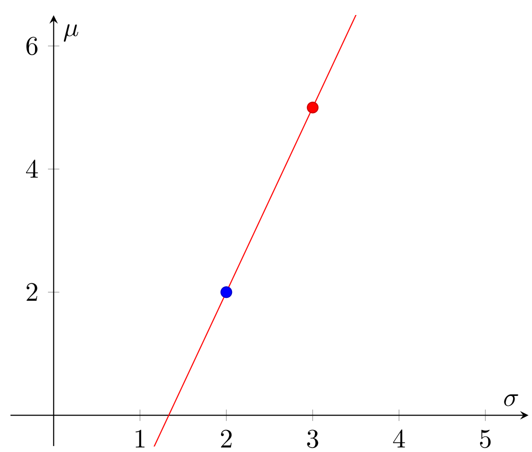
Then the variance is

$$\sigma_c^2 = \alpha^2\sigma_1^2 + (1-\alpha)\sigma_2^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2.$$

If you stare at this equation for a few minutes and think back to your high school math class, you'll notice that this is a perfect square. This means that you can simplify the equation.

$$\begin{aligned}&\sigma_c^2 = (\alpha\sigma_1 + (1-\alpha)\sigma_2)^2 \\ \Rightarrow \quad &\sigma_c = \alpha\sigma_1 + (1-\alpha)\sigma_2\end{aligned}$$

We can see here that the standard deviation is a linear function of α. We know from earlier that the expected return is always a linear function of α. This means that if we plot the possible portfolios in mean-standard deviation space, it will just be a straight line through the two assets.



Now let's consider another special case. Since we did the worst case scenario, let's now do the best case scenario. Suppose that $\rho = -1$. The variance formula now becomes

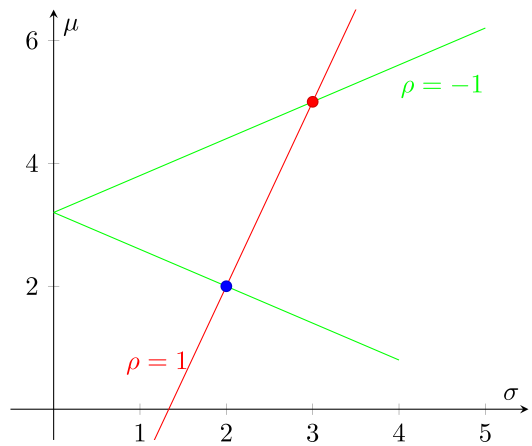
$$\sigma_c^2 = \alpha^2\sigma_1^2 + (1-\alpha)\sigma_2^2 - 2\alpha(1-\alpha)\sigma_1\sigma_2.$$

You probably noticed much more quickly this time that this is also a perfect square.

$$\begin{aligned}&\sigma_c^2 = (\alpha\sigma_1 - (1-\alpha)\sigma_2)^2 \\ \Rightarrow \quad &\sigma_c = \|\alpha\sigma_1 - (1-\alpha)\sigma_2\|\end{aligned}$$

This is also linear (at least piecewise) in α but it looks a bit different.

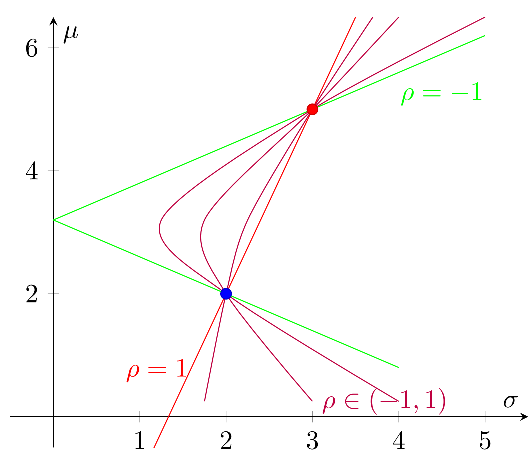
Notice that if we take
$$\alpha = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$
we can acheive zero variance. The possible portfolios are represented as the green line.



Now let us consider any other level for $\rho \in (-1, 1)$. Notice that while the expected return is a linear function in α, the standard deviation is not.
$$\sigma_c = \sqrt{\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2\rho}$$
It is more like a sideways parabola.

We can find farthest left point on the curve by finding the minimum variance portfolio.
$$\min_{\alpha} \quad \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2\rho$$
We do this by differentiating and setting it equal to zero. The derivative is,
$$\frac{\partial V}{\partial \alpha} = 2\sigma_1^2\alpha - 2\sigma_2^2(1-\alpha) + 2\sigma_1\sigma_2\rho - 4\sigma_1\sigma_2\rho\alpha.$$
Setting it equal to zero.
$$(2\sigma_1^2 + 2\sigma_2^2 - 4\sigma_1\sigma_2\rho)\alpha^* = 2\sigma_2^2 - 2\sigma_1\sigma_2\rho$$
$$\Rightarrow \quad \alpha^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$$

We can now draw this graph in mean-standard deviation space.



A key takeaway from this is that a mixture of the two assets can often have a lower variance than either of the two assets, and can get you onto a higher indifference curve.

Now that we have characterized the set of available portfolios, we need to find the element in that set that maximizes the agent's utility. This is the point where the indifference curve is tangent. We can solve this by differentiating the utility function with respect to α and setting it equal to zero.

$$\max_{\alpha} \quad \alpha\mathbb{E}[r_1] + (1-\alpha)\mathbb{E}[r_2] - \frac{A}{2}\alpha^2\sigma_1^2 - \frac{A}{2}(1-\alpha)^2\sigma_2^2 - A\alpha(1-\alpha)\sigma_1\sigma_2\rho$$

The derivative,

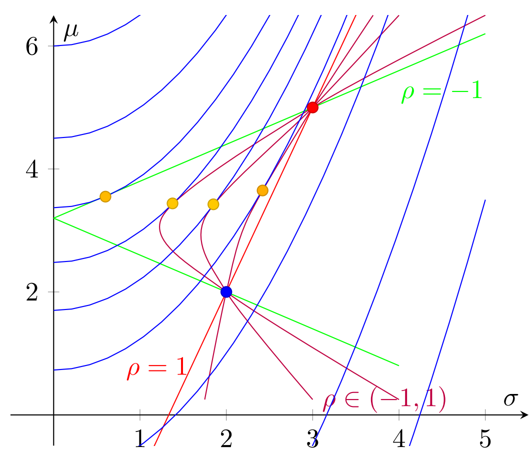
$$\mathbb{E}[r_1] - \mathbb{E}[r_2] - A\sigma_1^2\alpha + A(1-\alpha)\sigma_2^2 - A\sigma_1\sigma_2\rho + 2A\sigma_1\sigma_2\rho\alpha$$

Setting it equal to zero

$$\mathbb{E}[r_1] - \mathbb{E}[r_2] + A\sigma_2^2 - A\sigma_1\sigma_2 = A(\sigma_1^2 - \sigma_2^2 - 2\sigma_1\sigma_2)\alpha^*$$

Solving for α

$$\alpha^* = \frac{\mathbb{E}[r_1] - \mathbb{E}[r_2] + A\sigma_2^2 - A\sigma_1\sigma_2\rho}{A(\sigma_1^2 - \sigma_2^2 - 2\sigma_1\sigma_2\rho)}$$



This formula for α^* does not necessarily always lie between zero and one. A negative value of α wold represent short selling asset one to buy more of asset two.

4 Many Risky Assets

Suppose now that there are N risky assets with returns r_i. Let r denote the $N \times 1$ vector of returns. Similar to earlier, we will let α_i be the fraction of your wealth in asset i. α is the vector of α's. The return on your portfolio is then

$$r_p = \sum_{i=1}^{N} \alpha_i r_i = \alpha' r.$$

We can get the meaningful statistics of the portfolio return now. The expected return of your portfolio is

$$\mathbb{E}[r_p] = \sum_{i=1}^{N} \alpha_i \mathbb{E}[r_i] = \alpha'\mathbb{E}[r].$$

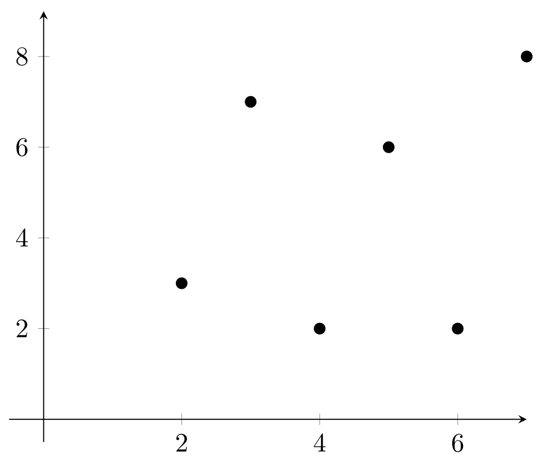
Let σ_{ij}^2 be the covariance between assets i and j. Notice that $\sigma_{ii}^2 = \sigma_i^2$. Then, let's let Σ be the variance-covariance matrix of r. That is

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 & \dots & \sigma_{1N}^2 \\ \sigma_{21}^2 & \sigma_2^2 & \dots & \sigma_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1}^2 & \sigma_{N2}^2 & \dots & \sigma_N^2 \end{bmatrix}.$$

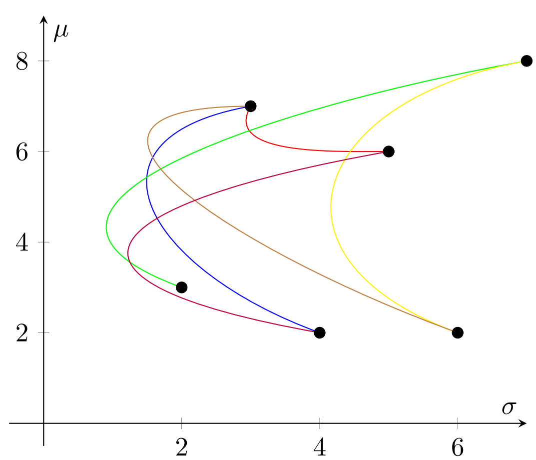
Now we can compute the variance of our portfolio.

$$\begin{aligned} \sigma_p^2 &= \alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + \ldots + \alpha_N^2\sigma_N^2 + 2\alpha_1\alpha_2\sigma_{12}^2 + 2\alpha_1\alpha_3\sigma_{13}^2 + \ldots + 2\alpha_{N-1}\alpha_N\sigma_{N-1,N}^2 \\ &= \sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j\sigma_{ij}^2 \\ &= \alpha'\Sigma\alpha \end{aligned}$$

Now let's look at the N assets graphed in mean-standard deviation space.

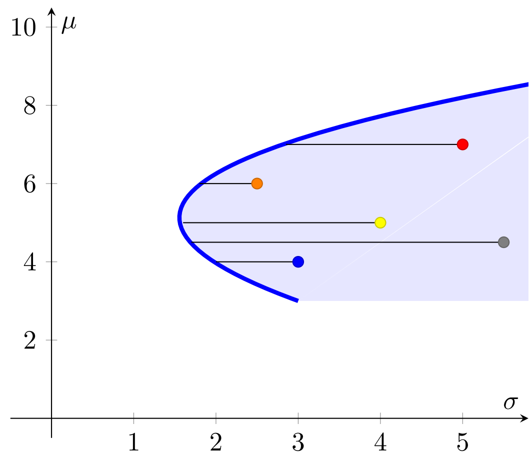


We know how to construct the portfolios from any two of the assets. A few of them are drawn below.



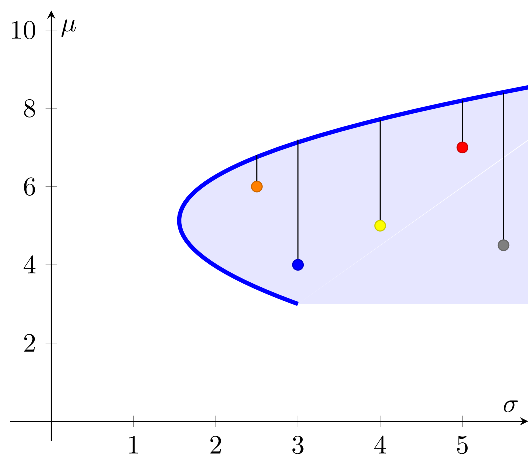
We can also construct portfolios of any three or more assets. In general, the shapes will look about the same. Each will still be a cartoon nose shape.

When there are many assets, we end up filling out an entire region on the graph. However, not the entire region is entirely intresting. Suppose you wanted an expected return of 3%. It is feasible for you to obtain this with a standard deviation of about two or higher or any higher number. But, why would you ever want to have a higher standard deviation than necessary. If for evey level of expected return we plot the minimum standard deviation portfolio, this will trace out a single curve.



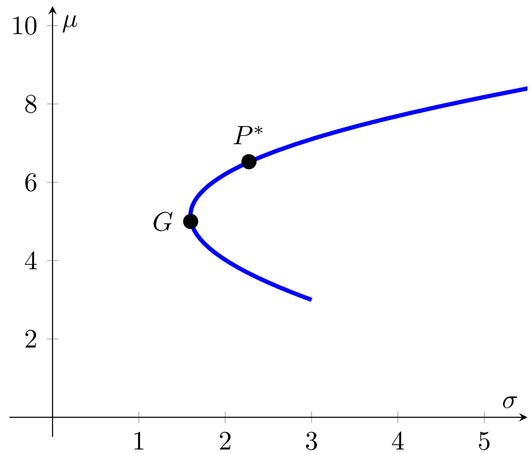
This curve is known as the **Efficient Frontier**. It is the lowest variance possible for any given expected return. You should think of it similarly to the production possibilities frontier. Any portfolio not on the frontier is wasteful, in the sense that it is strictly dominated by a portfolio that is on the frontier.

Alternatively, you could construct the frontier is the symmetric way. You could find the portfolio to maximize expected return subject to the constraint of a given standard deviation level.

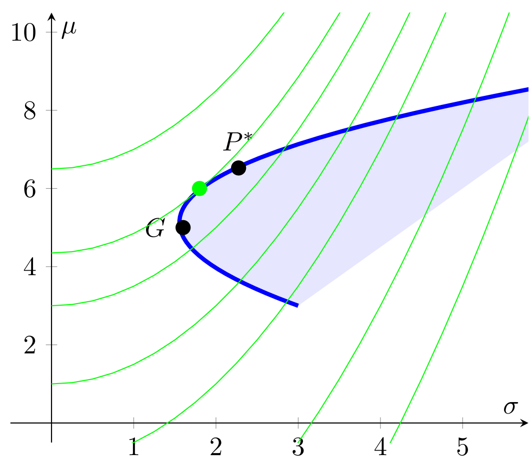


This will only give you the top half of the frontier in the picture here, but that is the half we care about anyway. The only problem with this second approach is that you need to know what the smallest σ you can use is. If you have something like above and you tell Excel to give you the highest expected return subject to the standard deviation being 1.0, Excel will likely just explode.

If we take the combination of any two portfolios on the frontier, the result will also be on the frontier. This leads to a third method we can use to obtain the frontier. First we ask our solver to find the minimum variance portfolio without constraining it to have any particular expected return. We will call this portfolio G. Next we take the risk free rate to be a fixed number, and we ask our solver to find the portfolio with the highest Sharpe ratio. We will call this portfolio P^*. Clearly, these two portfolios must be on the efficient frontier. Now we can trace out the entire frontier by simply taking all combinations of those two portfolios.



If there is no risk free asset available, we now maximize our utility subject to being on the frontier. This is done by finding the highest indifference curve that intersects the frontier.



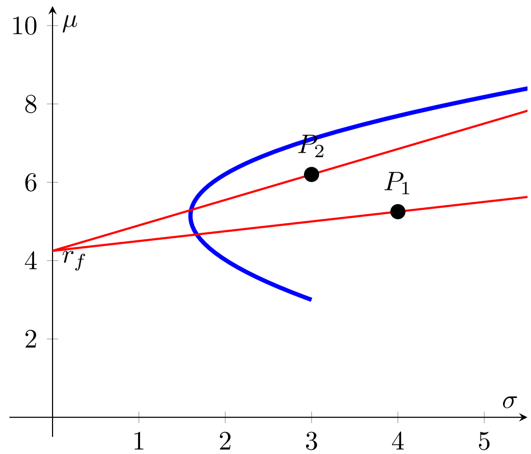
Which point on the frontier you choose will depend on your level of risk aversion. An agent that is more risk averse will choose a portfolio closer to G. An agent that is not very risk averse will choose a portfolio that is out past P^*. No agent, regardless of risk aversion, will choose a portfolio that is not on the frontier. Also, only the top half of the frontier will be chosen. There is no reason to choose a portfolio below G.

5 Many Risky Assets and a Risk Free Asset

Now let us suppose that you have many risky assets and one risk-free asset. Finally, we've made it to the case that reflects the real world (why not multiple risk-free assets?).

We can take inspiration for solving this from when we solved the problem of one risky asset and one risk free asset. We can treat any risky portfolio as the risky asset in that problem and solve for the optimum.

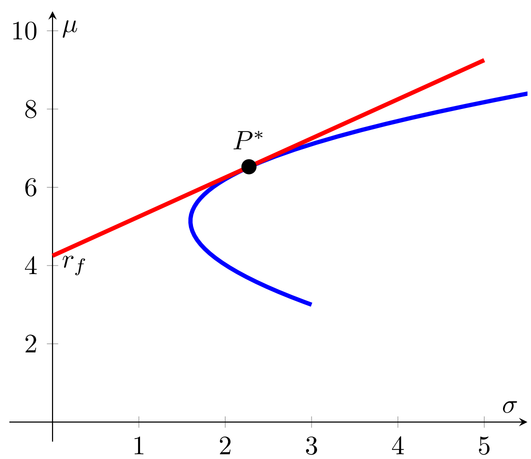
If we take P_1 and P_2 to be any portfolios constructed from the risky assets, we can draw the capital allocation line through the portfolio.



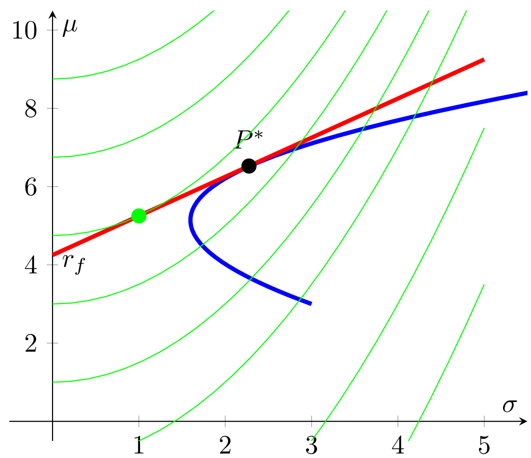
Any point on the lower red line can be obtained by putting some of your wealth in the risk free asset and some of it in portfolio P_1. However, notice that for any point on the lower line, there is a point on the line through P_2 that has a higher expected return with the same level of risk. Thus, for any level of risk aversion and any combination of P_1 and r_f, there is a portfolio you can construct from P_2 and r_f that dominates it.

But why stop at P_2. You could draw another capital allocation line that is higher than P_2 and will dominate everything on that line. You can keep on moving the line up until you have the highest line that still intersects the efficient frontier.

Remember that we move the line up by increasing the slope, and the slope of the capital allocation line is the Sharpe ratio. So, this is the same as saying that we want to find the risky portfolio that has the highest Sharpe ratio and draw our capital allocation line through it. This is the P^* portfolio that we found earlier, and the line will be just tangent to the efficient frontier.



The problem is then simply to maximize our utility subject to being on the red line. Let's again draw the indifference curves.



Every investor's optimal allocation will be somewhere on that red line. It will be closer to r_f if they are more risk averse, and it could even be beyond P^* if they are not very risk averse (this means you are borrowing at rate r_f to invest more in P^*). This means that everyone ought to be putting their wealth in only r_f and P^*. There need only exists two funds in the world. One money market fund will exist to give you a risk free return, and one risky fund that simply holds P^*. There is no reason to invest any money in anything else.

Why do you think there are so many other investment options in the world? Why do people invest in them?

Now, let's do some math to understand more about the optimal allocations. I will let $\underline{1}$ denote a vector that has only 1's in it. Let α be the vector that has the fraction of your wealth you put into each asset in the return vector r. Now we can maximize your utility function. Using the equations at the begining of the last section, the problem becomes the following.

$$\max_{\alpha \in \mathbb{R}^N} \quad r_f + \alpha' \left(\mathbb{E}[r] - r_f \underline{1}\right) - \frac{A}{2}\alpha' \Sigma \alpha$$

We can solve this for the optimal α vector by differentiating this with respect to α and setting it equal to zero. The derivative is

$$\frac{\partial U}{\partial \alpha} = \mathbb{E}[r] - r_f \underline{1} - A\Sigma\alpha.$$

Now we can set it equal to zero and solve for α^*.

$$\alpha^* = \frac{1}{A}\Sigma^{-1}\left(\mathbb{E}[r] - r_f \underline{1}\right)$$

We know that the matrix Σ is invertible because it is a variance-covariance matrix. This equation is clearly just the matrix equivalent of what we got in the case of one risky asset and one risk-free asset.

Let's now try writing out the problem without any vectors and see if we can solve any of it. We need to choose N different α's to represent the fraction of wealth in each of the N assets. Using the equations at the beginning of the last section, the problem becomes,

$$\max_{\alpha_1, \alpha_2, \ldots, \alpha_N} \quad r_f + \sum_{i=1}^{N} \alpha_i \left(\mathbb{E}[r_i] - r_f\right) - \frac{A}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j \sigma_{ij}^2.$$

We need to differentiate this equation with respect to each of the N α's and set all of them equal to zero. Then we have a system of N equations and N unknowns. We can solve this to get the optimal portfolio.

Let's start with differentiating with respect to α_1.

$$\frac{\partial U}{\partial \alpha_1} = \mathbb{E}[r_1] - r_f - A\sum_{i=1}^{N} \alpha_i \sigma_{i1}^2$$

Setting this equal to zero and solving for α_1^* gives

$$\alpha_1^* = \frac{\mathbb{E}[r_1] - r_f - A\sum_{i=2}^{N}\alpha_i^*\sigma_{i1}^2}{A\sigma_1^2}.$$

You could do the exact same thing to get an equation for α_2^*, α_3^*, and so on for any of the α_i^*. They are all defined in terms of each other. Then, you need to solve the system of equations to get α_i^* in terms of primatives.

Instead of solving it out completely, let's back up a step and see if we can get some intuition. When we set the equation equal to zero we got

$$\mathbb{E}[r_i] - r_f = A\sum_{j=1}^{N}\alpha_j^*\sigma_{ij}^2.$$

Remember that covariance operators are linear. Notice the following.

$$\begin{aligned} Cov(r_i, r_p) &= Cov(r_i, \alpha_1 r_1 + \alpha_2 r_2 + \ldots + \alpha_N r_N) \\ &= Cov(r_1, \alpha_1 r_1) + Cov(r_1, \alpha_2 r_2) + \ldots + Cov(r_1, \alpha_N r_N) \\ &= \alpha_1\sigma_1^2 + \alpha_2\sigma_{12}^2 + \ldots + \alpha_N\sigma_{1N}^2 \end{aligned}$$

Now we can clearly see that the optimality equation above becomes

$$\mathbb{E}[r_i] - r_f = A\sigma_{ip}^2.$$

This equation has the same interpretation as every maximization problem you did in 3101. We are setting the marginal benefit equal to the marginal cost. The marginal benefit of holding a little bit more of asset i is the excess return you get on asset i. The marginal cost of asset i is the increase in risk.

This same equation holds for all asset's available. In other words, you need to set the ratio of marginal benefit to marginal cost equal for every asset. This implies,

$$\frac{\mathbb{E}[r_i] - r_f}{\sigma_{ip}} = \frac{\mathbb{E}[r_j] - r_f}{\sigma_{jp}} = A = \frac{\mathbb{E}[r_p] - r_f}{\sigma_p^2}.$$

The part that you may think is strange at first is that I am refering to σ_{ip}^2 as the risk of asset i. Isn't σ_i^2 the risk of asset i? In isolation it would appear so, but the true measure of risk isn't how volatile the return is. The true riskiness is how it relates to the rest of your portfolio.

To see this, think about an asset that is negatively correlated to your portfolio. If you were to buy it, the overall variance of the portfolio would go down, even though you are buying something with positive variance.

Think back to my example on the second day of class with the chocolates. The contract we made up on the board looked risky in isolation, but really it was used to reduce risk.

Now, how do we equate the ratios of excess return to risk across assets? Imagine that asset i had a higher ratio than asset j.

$$\frac{\mathbb{E}[r_i] - r_f}{\sigma_{ip}} > \frac{\mathbb{E}[r_j] - r_f}{\sigma_{jp}}$$

This means that asset i has a higher expected excess return, relative to its risk, than asset j does. Clearly you would like to buy more of asset i and less of asset j. As you buy more of asset i, it makes up a larger share of your overall portfolio. Thus, the return of the overall portfolio is more heavily affected by the return of asset i. That is to say, the correlation between the portfolio and asset i has increased. This decreases the fraction on the left hand side. Similarly, buying less of asset j increases the fraction on the right hand side. This process will continue until the two are equated.

You will continue to rebalance until you have equated this ratio for every pair of asset available.

This is everything that I have to say about modern portfolio theory. Tune in next time to learn about factor models.